AN ALGORITHM FOR THE TRAVELING SALESMAN PROBLEM[1]

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A ‘branch and bound’ algorithm is presented for solving the traveling salesman problem. The set of all tours (feasible solutions) is broken up into increasingly small subsets by a procedure called branching. For each subset a lower bound on the length of the tours therein is calculated. Eventually, a subset is found that contains a single tour whose length is less than or equal to some lower bound for every tour. The motivation of the branching and the calculation of the lower bounds are based on ideas frequently used in solving assignment problems. Computationally, the algorithm extends the size of problem that can reasonably be solved without using methods special to the particular problem.

1. Introduction

The traveling salesman problem is easy to state: A salesman, starting in one city, wishes to visit each of n-1 other cities once and only once and return to the start. In what order should he visit the cities to minimize the total distance traveled? For 'distance' we can substitute time, cost, or other measure of effectiveness as desired. Distance or costs between all city pairs are presumed known.

The problem has become famous because it combines ease of statement with difficulty of solution. The difficulty is entirely computational, since a solution obviously exists. There are (n-1)! possible tours, one or more of which must give minimum cost. (The minimum cost

[1] This is a refurbished edition of Little et al.'s 1963 paper. Please refer to note on the last page.

could conceivably be infinite – it is conventional to assign an infinite cost to travel between city pairs which have no direct connection.)

The traveling salesman problem recently achieved national prominence when a soap company used it as the basis of a promotional contest. Prizes up to $10,000 were awarded for finding the most correct links in a particular 33-city problem. Quite a few people found the best tour. (The tie breaking contest for these succesful mathematicians was to complete a statement of 25 words or less on "I like ... because ...".) A number of people, perhaps a little over-educated, wrote the company that the problem was impossible – an interesting misinterpretation of the state of the art.

For the early history of the problem, see Flood[1]. In recent years a number of methods for solving the problem have been put forward. Some suffer from inefficiency, others produce solutions that are not necessarily optimal, and still others require intuitive judgments that would be hard to program on a computer. For a detailed discussion, see Gonzalez[2]. We shall restrict our discussion to methods that (1) guarantee optimality, (2) seem reasonable to program and (3) are general, i.e. not *ad hoc* to the specific numerical problem.

Among such methods the approach that has been carried furthest computationally is that of dynamic programming. Held and Karp[3] and Gonzales[2] have independently applied the method and have solved various test problems on computers. Gonzalez programmed an IBM 1620 to handle problems up to 10 cities. In his work the time to solve a problem grew somewhat faster than exponentially as the number of cities increased. A 5-city problem took 10 seconds, a 10-city problem took 8 minutes, and the addition of one more city multiplied the time by a factor, which, by 10 cities, had grown to 3. Storage requirements expanded with similar rapidity.

Held and Karp[3] have solved problems up to 13 cities by dynamic programming using an IBM 7090. A 13-city problem required 17 seconds. But such is the power of an exponential that, if their computation grows at the same rate as that of Gonzalez, a 20-city problem would require about 10 hours. Storage requirements, however, may become prohibitive before then. For larger problems than 13 cities, Held and Karp develop an approximation that seems to work well but does not guarantee an optimal tour.

We have found two papers in which the problem has been approached by methods similar to our 'branch and bound' algorithm. Rossman, Twery and Stone[4] in an unpublished paper apply ideas which they have called combinatorial programming[5]. To illustrate their method they present a 13-city problem. It was solved in 8 man-days. We have solved their problem by hand in about 3½ hours. Eastman[6], in an unpublished doctoral thesis and laboratory report, presents a method of solution and several variations on it. His work and ours contain strong similarities. However, to use our terminology, his ways of choosing branches and of calculating bounds are different from ours. He basically solves a sequence of assignment problems which give his bounds. We have a simpler method, and for branching we use a device which has quite a different motivation. The biggest problem Eastman solves is 10 cities and he gives no computation times, so that effective comparisons are difficult to make.

Most published problems are symmetric, i.e., the distance from city *i* to city *j* is the same as from *j* to *i*. The algorithm to be presented also works for asymmetric problems; in fact, it seems to work better. Asymmetric problems arise in various applications. As an example from production scheduling, suppose that there is a production cycle of some time period, during which an assembly line must produce each of *n* different models. The cost of switching from model *i* to model *j* is c_{ij}. What order of producing models minimizes total setup cost? This is a traveling salesman problem in which it would not necessarily be expected that $c_{ij} = c_{ji}$.

To summarize, 13 cities is the largest problem which we know about that has been solved by a general method which guarantees optimality and which can reasonably be programmed for a computer. Our method appreciably increases this number. However, the time required increases at least exponentially with the number of cities and eventually, of course, becomes prohibitive. Detailed results are given below.

2. The Algorithm

The basic method will be to break up the set of all tours into smaller and smaller subsets and to calculate for each of them a lower bound on the cost (length) of the best tour therein. The bounds guide the partitioning of the subsets and eventually identify an optimal tour - when a subset is found that contains a single tour whose cost is less than or equal to the lower bounds for all other subsets, that tour is optimal.

The subsets of tours are conveniently represented as the nodes of a tree and the process of partitioning as a branching of the tree. Hence we have called the method 'branch and bound'.

The algorithm will simultaneously be explained and illustrated by a numerical example. The explanation does not require reference to the example, however, for those readers who wish to skip it.

2.1 Notation

The costs of the traveling salesman form a matrix. Let the cities be indexed by $i = 1, \ldots , n$. The entry in row *i* and column *j* of the matrix is the cost for going from city *i* to city *j*. Let

$$C = [c(i,j)] = \text{cost matrix.}$$

C will start out as the original cost matrix of the problem but will undergo various transformations as the algorithm proceeds. A tour, *t*, can be represented as a set of *n* ordered city pairs, e.g.,

$$t = [\ (i_1, i_2)\ (i_2, i_3) \ldots (i_{n-1}, i_n)\ (i_n, i_1)\ ],$$

which form a circuit going to each city once and only once. Each *(i,j)* represents an arc or leg of the trip. The cost of a tour, *t*, under a matrix, *C*, is the sum of the matrix elements picked out by *t* and will be denoted by *z(t)*:

$$z(t) = \textstyle\sum_{(i,j)\ in\ t}\ c(i,j).$$

Notice that *t* always picks out one and only one cost in each row and each column. Also, let

X, Y, $\underline{Y}$ = nodes of the tree;
$w(X)$ = a lower bound on the cost of the tours of X, i.e., $z(t) \geq w(X)$ for t a tour of X;
z_0 = the cost of the best tour found so far in the algorithm.

2.2 Lower Bounds

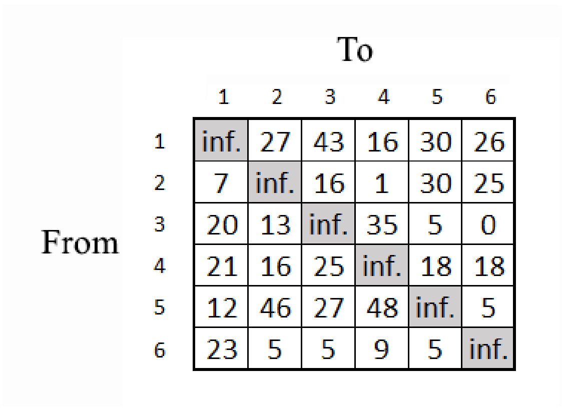
A useful concept in constructing lower bounds will be that of reduction. If a constant, *h*, is subtracted from each element of a row of the cost matrix, the cost of any tour under the new matrix is *h* less than under the old. This is because every tour must contain one and only one element from that row. The relative costs of all tours are unchanged, however, and so any tour optimal under the old will be optimal under the new.

The process of subtracting the smallest element of a row from each element in the row will be called *reducing* the row. A matrix with nonnegative elements and at least one zero in each row and column will be called a *reduced matrix* and may be obtained, for example, by reducing rows and columns. If $z(t)$ is the cost of a tour t under a matrix before reduction, $z_1(t)$ the cost under the matrix afterward, and h the sum of constants used in making the reduction, then

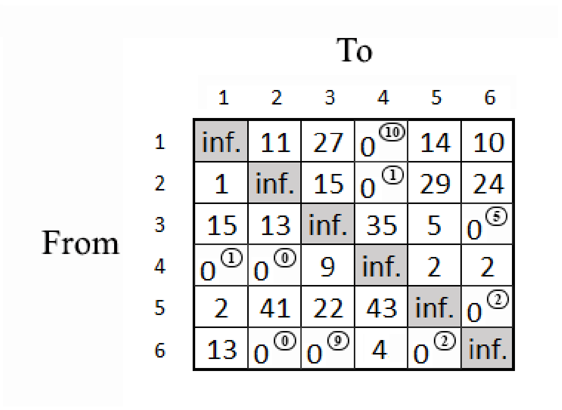
$$z(t) = h + z_1(t). \qquad (1)$$

Since a reduced matrix contains only nonnegative elements, *h* constitutes a lower bound on the cost *t* under the old matrix.

Consider then the 6-city problem shown in Fig. 1. Reduction of the matrix by rows, then columns, gives the matrix of Fig. 2. Total reduction is 48 so that $z(t) \geq 48$ for all t.



$$\text{From} \quad \begin{array}{c|cccccc} & \multicolumn{6}{c}{\text{To}} \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \text{inf.} & 27 & 43 & 16 & 30 & 26 \\ 2 & 7 & \text{inf.} & 16 & 1 & 30 & 25 \\ 3 & 20 & 13 & \text{inf.} & 35 & 5 & 0 \\ 4 & 21 & 16 & 25 & \text{inf.} & 18 & 18 \\ 5 & 12 & 46 & 27 & 48 & \text{inf.} & 5 \\ 6 & 23 & 5 & 5 & 9 & 5 & \text{inf.} \end{array}$$



$$\text{From} \quad \begin{array}{c|cccccc} & \multicolumn{6}{c}{\text{To}} \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \text{inf.} & 11 & 27 & 0^{\textcircled{10}} & 14 & 10 \\ 2 & 1 & \text{inf.} & 15 & 0^{\textcircled{1}} & 29 & 24 \\ 3 & 15 & 13 & \text{inf.} & 35 & 5 & 0^{\textcircled{5}} \\ 4 & 0^{\textcircled{1}} & 0^{\textcircled{0}} & 9 & \text{inf.} & 2 & 2 \\ 5 & 2 & 41 & 22 & 43 & \text{inf.} & 0^{\textcircled{2}} \\ 6 & 13 & 0^{\textcircled{0}} & 0^{\textcircled{9}} & 4 & 0^{\textcircled{2}} & \text{inf.} \end{array}$$

Fig.1. (left) Cost matrix for a 6-city problem. A typical tour might be t=[(1,3) (3,2) (2,5) (5,6) (6,4) (4,1)], which has the cost (length) z = 43+13+30+5+9+21=121. **Fig.2. (right)** Cost matrix after reducing rows and columns. Circled numbers are values of θ(*i,j*).

2.3 Branching

The splitting of the set of all tours into disjoint subsets will be represented by the branching of a tree, as illustrated bij Fig. 3. The node containing 'all tours' is self-explanatory. The node containing *i,j* represents all tours which include the city pair *(i,j)*. The node containing $\underline{i,j}$ represents all tours that do *not*. At the *i,j* node there is another branching. The node containing $\underline{k,l}$ represents all tours that include *(i,j)* but not *(k,l)*, whereas *k,l* represents all tours that include both *(i,j)* and *(k,l)*. In general, by tracing from a node, *X*, back to the start, we can pick up which city pairs are committed to appear in the tours of *X* and which are forbidden from appearing. If the branching process is carried far enough, some node will eventually represent a single tour. Notice that at any stage of the process, the union of the sets represented by the terminal nodes is the set of all tours.

When a node *X* branches into two further nodes, the node with the newly committed city pair will frequently be called *Y* and the node with the newly forbidden city pair $\underline{Y}$.

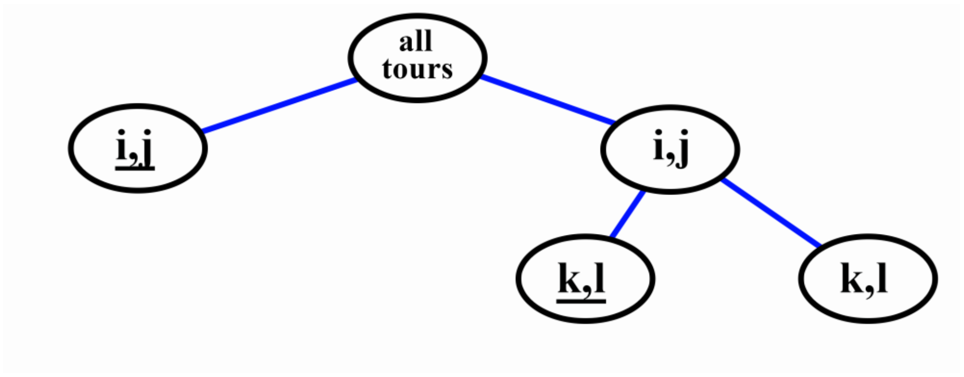


Fig. 3. Start of tree.

2.4 Flow Chart

The workings of the algorithm will be explained by tracing through the flow chart of Fig. 4.

Box 1 starts the calculation by putting the original cost matrix of the problem into *C*, setting *X=1* to represent the node, 'all tours', and setting the cost of the best tour so far to infinity.

Box 2 reduces the matrix and labels node *X* with its lower bound *w(X)*.

Box 3 selects *(k,l)*, the city pair on which to base the next branching. The goal in doing this is to split the tours of *X* into a subset (*Y*) that is quite likely to include the best tour of the node and another ($\underline{Y}$) that is quite unlikely to include it. Possible low cost tours to consider for *Y* are those involving an (*i,j*) for which c(*i,j*)=0.

Consider therefore the costs for tours that do not contain (*i,j*), i.e., possible tours for $\underline{Y}$. Since city *i* must be reached from *some* city, these tours must incur at least the cost of the smallest element in column *j*, excluding c(*i,j*). Since city *j* must connect to *some* city, the tours must incur at least the cost of the smallest element in row *i*, excluding *c*(*i,j*). Call the sum of these costs θ(*i,j*). We shall choose (k,l) to be that city pair that gives the largest θ(*i,j*). [This amounts to a search over *(i,j)* such that *c(i,j)* = 0, since otherwise θ(*i,j*) = 0.] Notice that, if *c(i,j)* is set to infinity and then row *i* and column *j* are reduced, the sum of the reducing constants is θ(*i,j*).

For the example, the θ(*i,j*) values are written in small circles placed placed in the cells of the zeroes of Fig.2. The largest θ is θ(1,4) = 10 + 0 = 10 and so (1,4) will be the first city pair used for branching.

Box 4 extends the tree from node *X* to $\underline{Y}$. As will be shown below, *w*($\underline{Y}$) = *w(X)* + θ(*k,l*). In the example *w*($\underline{Y}$) = 10 + 48 = 58 and the node is so labeled in Fig. 5b.

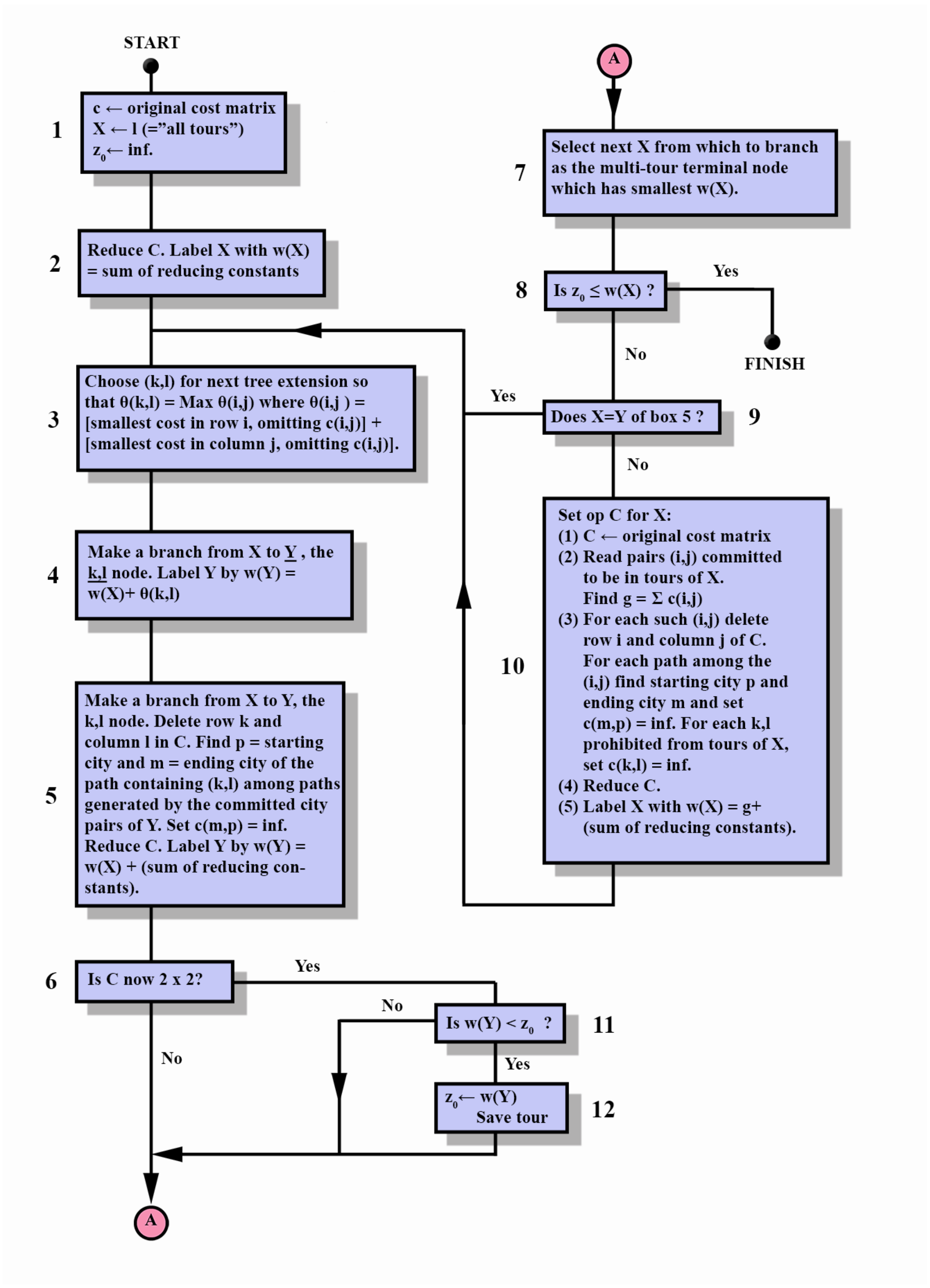

START
1
c ← original cost matrix
X ← I (="all tours")
z0 ← inf.
2
Reduce C. Label X with w(X) = sum of reducing constants
3
Choose (k,l) for next tree extension so that θ(k,l) = Max θ(i,j) where θ(i,j) = [smallest cost in row i, omitting c(i,j)] + [smallest cost in column j, omitting c(i,j)].
4
Make a branch from X to Y, the k,l node. Label Y by w(Y) = w(X)+ θ(k,l)
5
Make a branch from X to Y, the k,l node. Delete row k and column l in C. Find p = starting city and m = ending city of the path containing (k,l) among paths generated by the committed city pairs of Y. Set c(m,p) = inf. Reduce C. Label Y by w(Y) = w(X) + (sum of reducing constants).
6
Is C now 2 x 2?
Yes
No
No
11
Is w(Y) < z0 ?
Yes
12
z0 ← w(Y)
Save tour
A
A
7
Select next X from which to branch as the multi-tour terminal node which has smallest w(X).
8
Is z0 ≤ w(X) ?
Yes
FINISH
No
Yes
9
Does X=Y of box 5 ?
No
10
Set op C for X:
(1) C ← original cost matrix
(2) Read pairs (i,j) committed to be in tours of X. Find g = Σ c(i,j)
(3) For each such (i,j) delete row i and column j of C. For each path among the (i,j) find starting city p and ending city m and set c(m,p) = inf. For each k,l prohibited from tours of X, set c(k,l) = inf.
(4) Reduce C.
(5) Label X with w(X) = g+ (sum of reducing constants).

Fig. 4. Flow chart of the algorithm.

Box 5 sets up *Y*. Since the pair *(k,l)* is now committed to the tours, row *k* and column *l* are no longer needed and are deleted from *C*. Next, notice that *(k,l)* will be part of some connected path generated by the city pairs that have been committed to the tours of *Y*.

Suppose the path starts at city *p* and ends at city *m*. (Possibly $p=k$ or $m=l$ or both.) The connecting of *m* to *p* should be forbidden for it would create a subtour (a circuit with less than *n* cities) and no subtour can be part of a tour. Therefore, set $c(m,p) = inf$.

After these modifications *C* can perhaps be reduced in the following places: row *m*, column *p*, any columns that had a zero in row *k*, and any rows that had a zero in column *l*. All other rows and columns contain some zero that cannot have been disturbed. Let *h* be the sum of the new reducing constants. The lower bound for Y will now be shown to be

$$w(Y) = w(X) + h$$

The algorithm operates so that the investigation of each node, *X*, starts in Box 3 with a matrix *C* and a lower bound $w(X)$ that stand in a special relation. If *t* is any tour of *X*, $z(t)$ its cost under the original matrix, t_1, the city pairs of *t* left after removing those committed to the tours of *X*, and $z_1(t_1)$ the cost of t_1 under *C*, then it will be shown that

$$z(t) = w(X) + z_1(t_1). \qquad (2)$$

This expression is true for the first node by (1). Suppose that from a bound $w(X_1)$ and matrix C_1 of a node X_1, the algorithm constructs a bound $w(X_2)$ and reduced matrix C_2 for a node X_2. (X_2 will be on some branch out of X_1.) It will be shown that, if (2) is true for X_1, (2) will also be true for X_2.

The operations on C_1 to get C_2 (shown in Boxes 5 and 10) are always of the form: delete row *i* and column *j* for each (i,j) committed to the tours of X_2, insert various infinities, reduce. The lower bound is always of the form

$$w(X_2) = w(X_1) + \Sigma\, c_1(i,j) + h, \qquad (3)$$

where the summation is over the city pairs committed in X_2 but not in X_1 and *h* is the sum of the reducing constants. But consider any *t* in X_2 (and therefore in X_1). If we let $z_1(t_1)$ be the cost of the uncommitted city pairs of X_1 under C_1 and $z_2(t_2)$ be the cost of the uncommitted city pairs of X_2 under C_2,

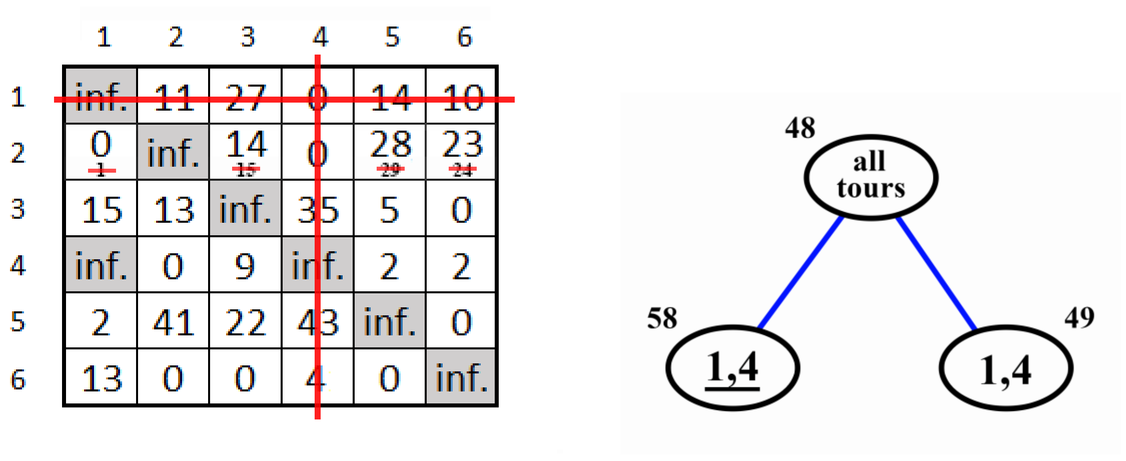
$$z_1(t_1) = \Sigma\, c_1(i,j) + h + z_2(t_2)$$

or using (2), assumed true for X_1,

$$z(t) = w(X_1) + \Sigma\, c_1(i,j) + h + z_2(t_2)$$

$$= w(X_2) + z_2(t_2)$$

so that (2) is true for X_2, as was to be shown.
Equation (3) is used to calculate the lower bounds in Boxes 4, 5, and 10. That these lower bounds are valid is established by (2) and the nonnegativity of the elements of *C*.



	1	2	3	4	5	6
1	inf.	11	27	0	14	10
2	0 ~~1~~	inf.	14 ~~15~~	0	28 ~~29~~	23 ~~24~~
3	15	13	inf.	35	5	0
4	inf.	0	9	inf.	2	2
5	2	41	22	43	inf.	0
6	13	0	0	4	0	inf.

Fig. 5. (left) Matrix after deletion of row 1 and column 4. (right) First branching.

For the example, the matrix of Fig.5 shows the deletion of row 1 and column 4. The connected path containing (1,4) is (1,4) itself, so that (*m*,*p*) = (4,1) and we set *c*(4,1) = *inf*. Looking for reductions, we find that row 2 can be reduced by 1. The *w*(*Y*) = 48+1 = 49 as shown.

It may be worth giving another example of finding (*m*,*p*). Suppose the committed city pairs were (2,1) (1,4) (4,3) and (5,6) and (k,l) were (1,4). Then the connected path containing (*k*,*l*) would start at 2 and end at 3 to yield (*m*,*p*) = (3,2).

Box 6 checks to see whether a single tour node is near.

Box 7 selects the next *node* for branching. There are a number of ways the choice might be made. The way shown here is to pick the node with the smallest lower bound. This leads to the fewest nodes in the tree.

Box 8 checks to see whether the algorithm is finished – whether the best tour so far has a cost less than or equal to the lower bounds on all terminal nodes of the tree.

Box 9 is a time saver. Most branching is from *Y* nodes, i.e., to the right. Such branching involves crossing out rows and columns and other manipulations that can be done on the matrix left over from the previous branching. When this case occurs, Box 9 detects it and the algorithm returns directly to Box 3.

Box 10 takes up the alternate case of setting up an appropriate lower bound and reduced matrix for an arbitrary *X*. Starting from the original cost matrix, rows and columns are deleted for city pairs committed to the tours of *X*, infinities are placed to block subtours and at forbidden city pairs, and the resulting matrix is reduced. The lower bound can be computed from (3) by thinking of X_1 in (3) as a starting node with $w(X_1) = 0$ and matrix equal the original cost matrix. Since different ways of reducing a matrix may lead to different sums for the reducing constants, the recalculated *w(X)* is substituted for the former one.

Boxes 11 and 12 finish up a single tour node. By the time *C* is a 2x2 matrix, there are only two feasible *(i,j)* left and they complete a tour. Since the box is entered with a reduced matrix, the costs of the final commitments are zero and $z=w(Y)$ by (2). If $z<z_0$, the new tour is the best yet and is read off the tree to be saved.

Returning to the example, Box 7 picks 1,4 as the second node for branching and, since this is a branching to the right, *C* is already available in reduced form. As shown in Fig.6, the next branching is on the basis of (2,1) with (*m*,*p*) = (4,2). Next, we go to the right from 2,1 on

(5,6) with (*m,p*) = (6,5) and then from 5,6 on the basis of (3,5) with (*m,p*) = (6,3). At this point C is a 2x2 matrix and we jump to Box 11 to finish the tour. We find z=63 which is stored as z_0 but, on returning to Boxes 7 and 8, we see that $\underline{1,4}$ has a lower bound of 58. To set up this node we go through Box 10. After the next branching, however, Box 8 shows that the problem is finished.

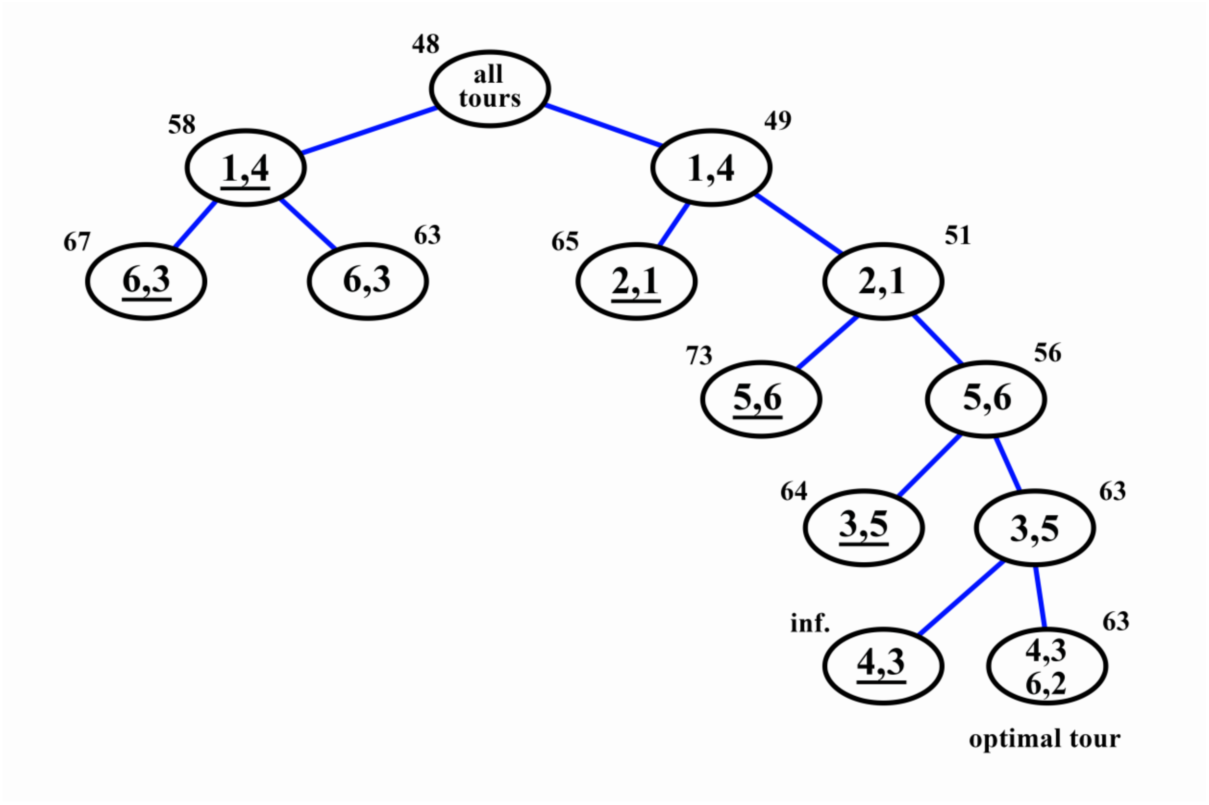


Fig. 6. Final Tree.

3. Discussion

At this point, let us stand back and review the general motivation of the algorithm. It proceeds by branching, crossing out a row and column, blocking a subtour, reducing the cost matrix to set a lower bound and then repeating. Although it is clear that the optimal solution will eventually be found, why should these particular steps be expected to be efficient? First of all, the reduction procedure is an efficient way of building up lower bounds and also of evoking likely city pairs to put into the tour. Branching is done so as to maximize the lower bound on the $\underline{k,l}$ node without worrying too much about the *k,l* node. The reasoning here is that the *k,l* node represents a smaller problem, one with the *k*th row and *l*th column crossed out. By putting the emphasis on a large lower bound for the larger problem, non-optimal tours are ruled out faster.

Insight into the operation of the algorithm is gained by observing that the crossing out of a row and column and the blocking of the corresponding subtour creates a new traveling salesman problem having one fewer city. Using the notation of Box 5, we can think of city *m* and city *p* as coalesced into a single city, say, *m'*. Setting c(*m,p*) = *inf.* is the same as setting c(*m', m'*) = *inf.* The blocking of subtours is a way of introducing the tour restrictions into what is otherwise an assignment problem and is accomplished rather successfully by the algorithm.

Finally, unlike most mathematical programming algorithms, the one here has an extensive memory. It is not required that a trial solution at any stage be converted into a new and better trial solution at the next stage. A trial branch can be dropped for a moment while another branch is investigated. For this reason there is considerable room for experiment in how the next branch is chosen. On the other hand the same property leads to the ultimate demise of the computation – for *n* sufficiently large there are just too many branches to investigate and a small increase in *n* is likely to lead to a large number of new nodes that require investigation.

4. Modifications

A variety of embellishments on the basic method can be proposed. We record several that are incorporated in the computer program used in later calculations. For the program itself, see Sweeney[9].

4.1 Go To The Right

It is computationally advantageous to keep branching to the right until it becomes obviously unwise. Specifically, the program always branches from the *k,l* node unless its lower bound exceeds or equals the cost of a known tour. As a result a few extra nodes may be examined, but usually there will be substantial reduction in the number of time-consuming setups of Box 10.

One consequence of the modification is that the calculation goes directly to a tour at the beginning. Then, if the calculations are stopped before optimality is proven, a good tour is available. There is also available a lower bound on the optimal tour. The bound may be valuable in deciding whether the tour is sufficiently good for some practical purpose.

4.2 Throw Away the Tree

A large problem may involve thousands of nodes and exceed the capacity of high-speed storage. Storage can be saved, although usually at the expense of time, by noting that, at any point in the computation, the cost of the best tour so far sets an upper bound on the cost of an optimal tour. Let the calculation proceed by branching to the right (storing each terminal node) until a single tour is found with some cost, say, z_0. Normally, one would next find the terminal node with the smallest lower bound and branch from there. Instead, work back through the terminal nodes, starting from the single tour, and discard nodes from storage until one is found with a lower bound less than z_0. Then, branch again to the right all the way to a single tour or until the lower bound on some right-hand node builds to z_0. (If the branch goes to the end, a better tour may be found and z_0 assigned a new, lower value). Repeat the procedure: again work up the branch, discarding terminal nodes with bounds equal or greater than z_0 until the first one smaller is found; again branch to the right, etc.

The effect of the procedure is that very few nodes need be kept in storage – something on the order of a few *n*. These form an orderly sequence stretching from the current operating node directly back to the terminal node on the left-most branch out of 'all tours'.

As an illustration, consider the problem and tree of Figure 6. The computation would proceed by laying out in storage the nodes 4,1; $\underline{2,1}$; $\underline{5,6}$; and $\underline{3,5}$. At the next step we find a tour with z_0 = 63 and the obviously useless node $\underline{4,3}$. The tour is stored separately from the tree. Working up the branch, first $\underline{3,5}$ is discarded, then $\underline{5,6}$ and $\underline{2,1}$, but $\underline{1,4}$ has a bound less than z_0. Therefore, branching begins again from there. A node $\underline{6,3}$ is stored and then we find the node to the right has a bound equal (s) z_0 and may be discarded. Working back up the tree again, $\underline{6,3}$ is discarded and, since that was the only remaining terminal node, we are finished.

The procedure saves storage but increases computation time. If the first run to the right turns up a rather poor tour, i.e. large z_0 tour, the criterion for throwing away nodes is too stiff. The calculation is forced to branch out from many nodes whose lower bounds actually exceed the cost of the optimal tour. The original method would never do this for it would never explore such nodes until it had finished exploring every node with a smaller bound. In the process, the optimal tour would be uncovered and so the nodes with larger bounds would never be examined.

4.3 Taking Advantage of Symmetry

If the travelling salesman problem is symmetric and *t* is any tour, another tour with the same cost is obtained by traversing the circuit in the reverse direction. Probably the most promising way to handle this is to treat the city pair (*i,j*) as not being ordered. This leads naturally to a new and somewhat more powerful bounding procedure. Although the basic ideas are not changed much, considerable reprogramming is required. So far, we have not done it.

There is another way to take advantage of symmetry and this one is easy to incorporate into our program. All reverse tours can be prohibited by modifying the nodes along the leftmost branch of the tree. These are the nodes with no city pairs committed but some forbidden. Suppose that such a node, X, branches into nodes, Y, with (*k,l*) committed, and, $\underline{Y}$, (k,l) forbidden. The reverse tours of Y all have (*l,k*) in them. They cannot be in Y for the presence of both (*k,l*) and (*l,k*) is not possible in any tour. Such of the reverse tours as were in X are in $\underline{Y}$. We may prohibit them by setting c(*l,k*) = *inf.* [as well as c(*k,l*) = *inf.*] in any matrix for Y. Thus, a reverse tour is prohibited as soon as the tour itself is identified to the extent of having one committed city pair.

4.4 A Computational Aid

In both hand and machine computation θ(*k,l*) is easiest calculated by first finding, for each row *k* and column *l* of the reduced matrix:

α(*k*) = the second smallest cost in row *k*.

β(*l*) = the second smallest cost in column *l*.

Then θ(*k,l*) = α(*k*) +β(*l*) for any (*k,l*) which has c(*k,l*) = 0. In a hand computation the α(*k*) can be written as an extra column to the right of the matrix and the β(*l*) as an extra row at the bottom. After working out a few problems, one can see that when the branching is to the right there is no need to search the whole matrix to reset *α* and *β* but that only certain rows and columns need be examined.

4.5 Other possibilities

If desired, the algorithm can be modified so as to generate all optimal solutions. Instead of discarding nodes with $w(X) = z_0$, split them up further until eventually all the terminal nodes either have $w>z_0$ or are optimal single tours with $z=z_0$. Our computer program does not include this modification because in some cases it will increase the computing time a great deal – suppose the cost matrix were all zeroes.

Quite possibly, the average computing time can be decreased by solving the assignment problem for the original cost matrix and reducing the matrix by the cost of the optimal assignment in Box 2. (Some methods for solving the assignment problem leave it in reduced form.) The advantage lies in the larger lower bound with which the problem starts. The closer the starting lower bound to the cost of the optimal tour, the less is the branching that may be expected. Our exploration of the possible gains has not been extensive and has yielded mixed results: Croes'20-city problem[8] was speeded up, but some others were lengthened.

The idea that we are calling 'branch and bound' is more general than the traveling salesman algorithm. A minimal solution for a problem can be found by taking the set of all feasible solutions, splitting it up into disjoint subsets, finding lower bounds on the objective function for each subset, splitting again the subset with the smallest lower bound, and so forth, until an optimal solution is found. The efficiency of the process however, rests very strongly on the devices used to split the subsets and to find the lower bounds. As a simple example of another use of the method, if the step of setting c(*m*,*p*) = *inf.* is omitted from the traveling salesman algorithm, it solves the assignment problem. For another example, see Doig and Land [10].

TABLE I

Mean and Standard Deviation of *T* for Random Distance Matrices

(T = time in minutes used to to solve Traveling Salesman Problem on IBM 7090.)

Number of cities	Number of problems solved	Mean T	Std. dev. T	Mean log T (a)	Std. dev. Log T (a)
10	100	0.012	0.007	log 0.015	log 1.24
20	100	0.084	0.063	log 0.067	log 2.09
30	100	0.975	1.24	log 0.63	log 2.94
40	5	8.37	10.2	log 4.55	log 3.74

(a) Obtained by plotting the cumulative frequency on log normal probability paper and fitting a straight line, except in the 40-cities case for which the computation was numerical. In case of 10 cities the log normal fits only the tail of the distribution – 30 per cent of the problems went directly to the solution without extra branching and thereby produced a lump of probability at 0.002 minute.

5. Calculations

Problems up to 10 cities can be solved easily by hand. Although we have made no special study of the time required for hand computations, our experience is that a 10-city problem can be solved in less than an hour.

The principal testing of the algorithm has been by machine on an IBM 7090. Two types of problems have been studied: (1) asymmetric distance matrices with elements consisting of uniformly distributed 3-digit random numbers and (2) various published problems and subproblems constructed therefrom by deleting cities. Most of the published problems have been made up from road atlases or maps and are symmetric.

The random distance matrices have the advantage of being statistically well defined. Average computing times are displayed in Table I and curve (a) of Fig. 7. Problems up to 20 cities usually require only a few seconds. The time grows exponentially, however, and by 40 cities is beginning to be appreciable, averaging a little over 8 minutes. As a rule of thumb, adding 10 cities to the problem multiplies the time by a factor of 10.

The standard deviation of the computing time also increases with problem size as may be seen in Table I. Because the distribution of times is skew, the simple standard deviation is a little misleading, at least for the purpose of estimating the probability of a long calculation. Consequently, a log normal distribution has been fitted to the tail of the distribution. A use of the tabulated numbers would be, for example, as follows: A two-sigma deviation on the high side in a 40-city problem would be a calculation which took $(3.74)^2(4.55) = 64$ minutes. In other words, the probability that a 40 city random distance problem will require 64 minutes or more is estimated to be 0.023.

Symmetric problems have usually taken considerably longer than random distance problems of the same size. To obtain a variety of problems of increasing size, we have taken published problems and abstracted subproblems of increasing size. The first 10 cities were taken, then the first 11 cities, etc., until the computing times become excessive. Curves (b) and (c) of Fig. 7 show the results for subproblems pulled out of the 25- and 48-city problems of Held and Karp[3]. The 25-city problem itself took 4.7 minutes. We note that Held and Karp's conjectured optimal solution is correct.

A few miscellaneous problems have also been solved. Croes'[8] 20-city problem took 0.126 minutes. A 64 'city' knight's tour took 0.178 minutes.

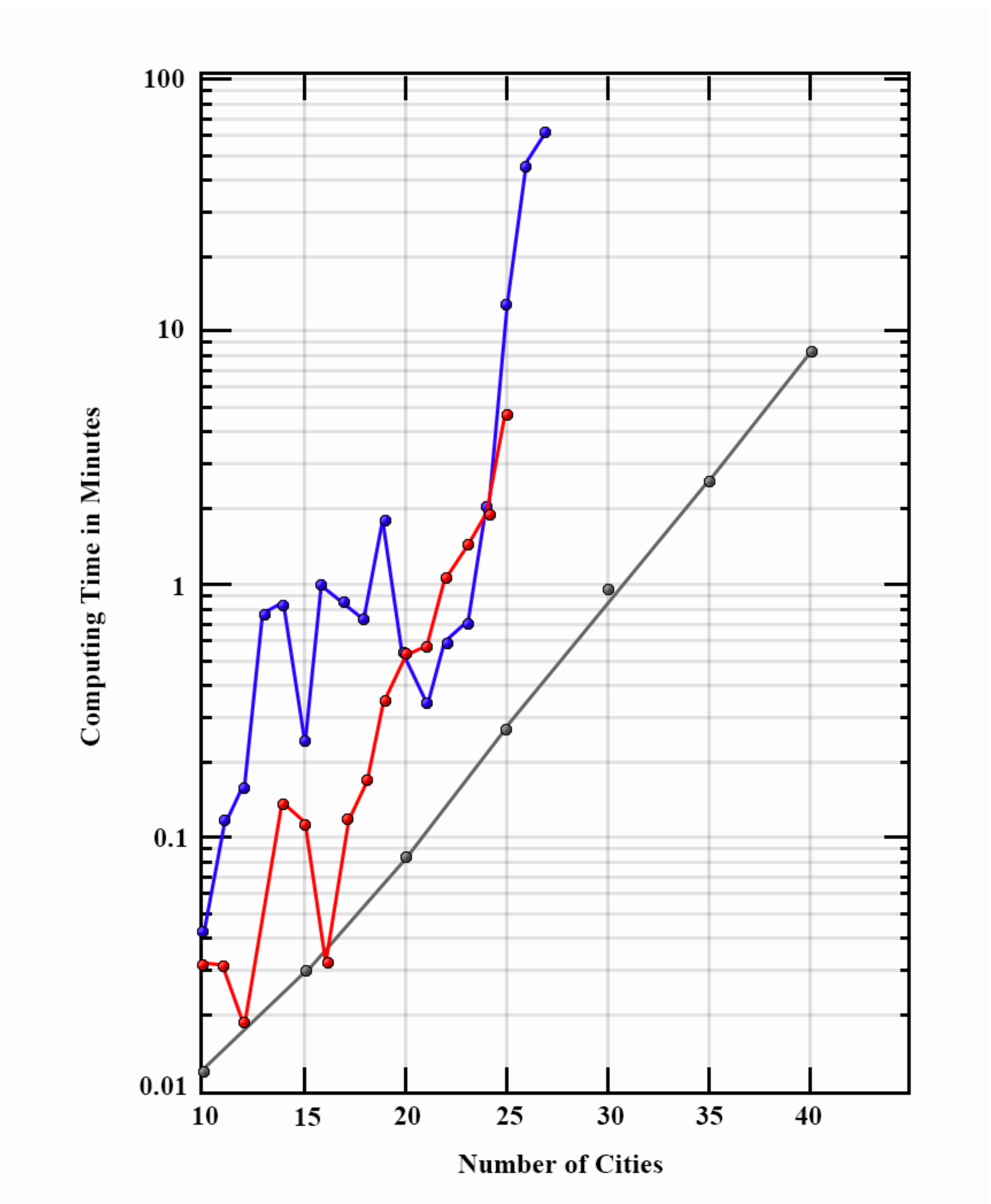


Fig. 7. Computing times on IBM 7090. Grey: Average times for 3 digit random number distance matrices. Red: Subproblems derived from Held and Karp's 25-city problem. Blue: Subproblems derived from Held and Karp's 48 city problem.

6. Acknowledgement

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Industrial Management Computing Facility. The time for the production runs was provided by IBM on the IBM7090 at the MIT Computation Center. We wish to acknowledge the help of R. H. Gonzalez for programming certain investigations. His time was supported by the US Army Research Office (Durham).

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8. Notes on this refurbished edition

Much inspired by Tim Rohlfs' refurbished edition of Stephen Cook's paper, I refurbished this paper by Little et al. (henceforth “Lital”), making it suitable for searching and editing, and hopefully adding some readability as well. I have almost literally stuck to the text as handed down to us by Little et al., but also made a few changes, mainly in layout, placing images and removing a few typo's. I also numbered the paragraphs, and a changelog is supplied after this note. At some point, I might turn it into a wiki-page or a more interactive format too.
Richard Olij did a tremendous job on proofreading the manuscript, but in case you find any typo's, or have other questions, please mail to Daanvandenberg1976 at the mailservice that starts with the seventh letter. Or look me up on LinkedIn, Facebook or Twitter.

9. Changelog

1. I've numbered the sections, which was not in the original paper, making it somewhat easier to discuss the paper (e.g.: over the phone).

2. The lower bound on the cost of tours in X, w(X), is it a w or an omega (ω)? For now, I chose a 'w'.

3. [typo] "cost under the original martix" => "cost under the original matrix"

4. Prizes up to $10,000 were [unreadable part] the most correct links in a particular 33-city problem => Prizes up to $10,000 were awarded for finding the most correct links in a particular 33-city problem

5. recently achieved national prom[unreadable part] => recently achieved national prominence

6. As will be shown below, [unreadable character]*(<u>Y</u>)* = *w(X)* + θ(*k,l*) => As will be shown below, w*(<u>Y</u>)* = *w(X)* + θ(*k,l*)

7. I replaced the overline by an underline. This does not have my preference, but it is simply much easier in Word.

8. I've redone figures 1 and 2, replacing infinity signs by "inf." and lowlightling them in grey. Values are the same, of course. I did put the pictures in a different place, directly below below the text that concerns them so the reader can 'follow' Lital's numerical example with even less effort than in the original paper. For this reason, words "left" and "right" were added in the caption.

9. In figure 5, I replaced "a" and "b" by "left" and "right"

10. In figure 7, I replaced letters with colours, also in the legend. The grey dot that appears to be missed by the line is also missed in the original paper.

11. and the node [unreadable] so labeled in Fig. 5b. => and the node is so labeled in Fig. 5b.

12. After consulting with colleagues, I replaced all infinity signs by "*inf.*". The reason is that the infinity signs simply don't turn out well in the rendered pdf. Maybe I'll find a solutiont for that later.

13. right has a bound equal ### z_0 and may be discarded => right has a bound equal z_0 and may be discarded (We're really not sure wat is says here. It looks like (s) or (8) but we're expecting 'to'.)